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Localized induction equation and pseudospherical surfaces

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Abstract. We describe a close connection between the localized induction equation hierarchy of integrable evolution equations on space curves and surfaces of constant negative Gauss curvature.

1. Introduction

Many of the integrable equations of nonlinear science have essentially equivalent realizations in terms of the classical geometry of curves and surfaces in space. These geometric realizations provide new insight into the structure of the integrable equations; in addition, these geometric problems may well have interesting physical interpretations in their own right. In this paper, we describe recent developments illustrating a close connection between two such geometric realizations: the localized induction equation (LIE) and pseudospherical surfaces or surfaces of constant negative Gauss curvature.

1.1. Localized induction equation

LIE is a local geometric evolution equation defined on space curves via the equation

$$\gamma_t = \gamma_s \times \gamma_{ss}$$

where s is the arclength parameter for the evolving space curve $\gamma(s, t) \in R^3$, and $\times$ denotes cross product. When the curvature is non-vanishing, the right-hand side can be written κB, where B is the binormal to γ, and κ is the curvature. LIE was developed in fluid mechanics as an idealized *local* model for the evolution of the centreline of a thin, isolated vortex tube in an inviscid fluid (for derivation and history, see [1–3]; for a discussion of more accurate, non-local models, see [4, 5]). As in the case of the full inviscid Euler equations from which it is derived, LIE can be described as a Hamiltonian evolution equation and, in fact, the corresponding Hamiltonian is just the length functional on space curves [6]. The connection of LIE to soliton theory was made apparent through a discovery of Hasimoto [7]: if γ evolves according to LIE, then the induced evolution of its *complex curvature* $\psi = \kappa \exp[\mathrm{i} \int^s \tau(u)\, \mathrm{d}u]$ (τ is the torsion along the curve) is given by the *cubic nonlinear Schrödinger equation* (NLS) $\psi_t = \mathrm{i}(\psi_{ss} + \frac{1}{2}|\psi|^2\psi)$. NLS is a well known example of a completely integrable evolution equation; Hasimoto's result implies that LIE is a geometric realization for NLS. Further investigations of the LIE–NLS correspondence were reported in [8, 9] and details of the complete integrability of LIE itself are also described there. We remark that the connection between NLS-type equations and the equations of fluid motion remains a topic of current research [10].

1.2. Pseudospherical surfaces

The study of pseudospherical surfaces in Euclidean space spans a period of more than a century; in particular, we mention the early works of Dobriner [11], Enneper [12], and Bäcklund [13]. Recent interest has been spurred by the connection with soliton theory [14, 15] (a kindred problem, finding metrics on R^2 with constant curvature, is also related to integrable evolution equations: see [16, 17]). We mention two such connections:

(i) Given a pseudospherical surface M, the angle ψ between its asymptotic curves satisfies the *sine–Gordon equation* (SG) $\partial^2\psi/\partial xy = \sin(\psi)$, where x and y are asymptotic coordinates for the surface (for basic definitions from surface theory, see [18]). Again, SG is a well known example of a completely integrable equation, which arises in numerous physical problems [19–21]. Thus, a pseudospherical surface M is a geometric realization of a given solution to SG.

(ii) Given a pseudospherical surface M, its second fundamental form induces a Lorentz metric on the surface. The Gauss map of M (taking M to the two sphere S^2) is a *harmonic map* [14, 15]. This is an example of a classical chiral model, for which there exists an extensive literature ([22–25] and references therein).

We now make a simple observation which demonstrates that LIE has *some* connection with surface theory. Consider any curve $\gamma = \gamma(s, 0)$ and let it evolve according to LIE. Because N is normal to the resulting swept-out surface, it follows that $\gamma(s, t)$ is a geodesic for any time t, thus providing a geodesic foliation of the resulting surface.

To describe the connection with pseudospherical surfaces, we make reference to the complete integrability properties of LIE. LIE is the first (non-trivial) term of an infinite sequence of commuting Hamiltonian evolution equations on curves, all of which equations are local-geometric in nature; we call this sequence the *localized induction hierarchy* (LIH). The associated Hamiltonians (which are conserved quantities for LIE) can be expressed as global geometric invariants of the curves. We shall see that certain distinguished *soliton curves* (= critical points for linear combinations of the Hamiltonians), after evolving according to a related linear combination of evolution equations from LIH, sweep out pseudospherical surfaces. In analogy with the geodesic construction of the previous paragraph, the induced foliation plays a role in the geometry of the surface: the curves of the foliation are asymptotic lines for the surface. The main point of this paper is to describe this construction. We also find an interesting connection between pseudospherical surfaces and Bäcklund transformations for certain curves; see section 4. In this same section, there is a surprising technical result suggesting deeper relations with Lie groups: two natural bases for a geometrically defined vector space, relevant to our theory, are related via a change-of-basis matrix defined in terms of lower triangular Toeplitz matrices. In the last section we discuss a related topic: evolution equations on surfaces which preserve the pseudosphericity property. For brevity, proofs have been omitted, but sufficient computational detail is presented so that the reader can at least reconstruct the basic examples described here.

One way of viewing our technique is as a 'nonlinear factorization' of the problem of constructing pseudospherical surfaces: the simpler 'factors' are the related variational problem on curves, and then the subsequent evolution of critical points of this variational problem according to appropriate evolution equations. Historically, we know that solution techniques for integrable systems 'travel well': if applicable to one integrable example, they can usually be modified to apply to essentially all other known integrable problems. Thus, this study of the LIE-pseudospherical connection will hopefully have consequences for the study of integrable models of more direct interest to mathematical physics.

2. LIH and related hierarchies

As stated above, LIE belongs to an infinite hierarchy of evolution equations on curves, all of the form $\gamma_t = X_n = aT + bN + cB$, where $\{T, N, B\}$ is the Frênet frame along the curve, and a, b, c are functions (polynomial) of $\kappa, \tau, \kappa' = \kappa_s, \tau' = \tau_s$, and higher derivatives with respect to s. We list the first few terms of the hierarchy, as well as their associated Hamiltonians (the vector field X_0 is exceptional):

$$X_0 = -T$$

$$X_1 = \kappa B \qquad I_1 = \int_\gamma \mathrm{d}s$$

$$X_2 = \tfrac{1}{2}\kappa^2 T + \kappa' N + \kappa\tau B \qquad I_2 = \int_\gamma -\tau\,\mathrm{d}s$$

$$X_3 = \kappa^2\tau T + (2\kappa'\tau + \kappa\tau')N + (\kappa\tau^2 - \kappa'' - \tfrac{1}{2}\kappa^3)B \qquad I_3 = \int_\gamma \frac{1}{2}\kappa^2\,\mathrm{d}s$$

$$X_4 = (-\kappa\kappa'' + \tfrac{1}{2}(\kappa')^2 + \tfrac{3}{2}\kappa^2\tau^2 - \tfrac{3}{8}\kappa^4)T + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \tfrac{3}{2}\kappa^2\kappa')N$$
$$+ (\kappa\tau^3 - 3(\kappa'\tau)' - \tfrac{3}{2}\kappa^3\tau - \kappa\tau'')B \qquad I_4 = \int_\gamma \frac{1}{2}\kappa^2\tau\,\mathrm{d}s$$

$$\ldots$$

The vector fields of LIH are *locally arclength preserving* (LAP): a vector field W is LAP if every segment of a curve γ has its length remain constant as γ evolves via $\gamma_t = W$. Equivalently, $\langle W_s, T\rangle = 0$.

The first few functionals in the list have simple physical interpretations. As shown in [26], the critical points of linear combinations of the functionals I_1, I_2, I_3 are the Kirchhoff rods of elasticity theory. Interestingly, these are exactly the curves for which the shape remains unchanged as they evolve according to LIE ([9, 27, 28]). Another discussion of the physical interpretation of the invariants of LIE can be found in [29].

As is usually the case with integrable systems, LIH is generated by a recursion operator $X_{n+1} = \mathcal{R}X_n$, $n \geqslant 0$; if $X = aT + bN + cB$ then $\mathcal{R}(X) = -\mathcal{P}(T \times X')$, where $\mathcal{P}$ is a *parameterization operator* $\mathcal{P}(X) = \int^s(\kappa b)\,\mathrm{d}s T + bN + cB$. Besides being useful for generating LIH, $\mathcal{R}$ can be used to compactly express the first-order variations in curvature and torsion along *any* vector field W which is LAP [9]:

$$W(\kappa) = \langle -\mathcal{R}^2(W), N\rangle \qquad W(\tau) = \langle -\mathcal{R}^2(W), B/\kappa\rangle'.$$

Related formulae also exist for the evolution of frame fields along W [30].

There are a number of hierarchies of integrable geometric evolution equations, related to LIE, which have interesting geometric properties. These are discussed in more detail in [31]; we mention those which are relevant here:

(i) *Constant torsion preserving (CTP)*: For $n \geqslant 0$, the vector fields

$$Z_n = \sum_{k=0}^{2n} \binom{2n+1}{k} (-\tau_0)^k X_{2n-k}$$

preserve the constant torsion condition $\tau = \tau_0$. If a constant torsion curve γ evolves according to $\gamma_t = Z_n$, the induced evolution on curvature $\kappa_t = Z_n(\kappa)$ is the corresponding element of the (mKdV) hierarchy; in particular, Z_1 induces the (mKdV) evolution $\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s$, recovering a result of Lamb [32]. Recently, Fukumoto and Miyazaki [33] have derived a refined version of LIE which allows for an axial velocity for the vortex tube: modulo trivial scaling terms, their equation is exactly $\gamma_t = Z_1$.

(ii) *Planar preserving*: A special case of (i) in section 1.2 is worth noting: when $\tau_0 = 0$, the sequence Z_n just reduces to the even X_{2n} restricted to planar curves. This integrable hierarchy of evolution equations has been discussed by several authors [34–36]. The first term of the hierarchy can be interpreted physically: in [37], it is shown that $\gamma_t = Z_1$, when restricted to planar curves, is a 'localized induction equation' for boundary curves of vortex patches for two-dimensional ideal fluid flow. The *even* functionals I_{2n} for LIH vanish identically on planar curves, and the *odd* functionals I_{2n+1} are restricted to give functionals on planar curves which depend only on κ and its derivatives.

(iii) *Torsion independent*: The vector fields $A_0 = -T$,

$$A_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (-\tau_0)^k X_{n-k} \qquad n \geqslant 1$$

have the property that, along curves γ with $\tau = \tau_0$, the coefficients of $A_n = aT + bN + cB$ have no explicit τ dependence. The odd vector fields in the sequence are purely binormal; the even vector fields, on the other hand, have a zero binormal component. We thus refer to the even fields as 'planar-like' and introduce the notation $\Omega_n = A_{2n}$.

3. Pseudospherical surfaces and the 'trigonometric equation'

We briefly review the basic facts from surface theory in R^3, mostly to establish notation and terminology. Given an oriented surface M, the *Gauss map* $\nu : M \to S^2$ sends a point $p \in M$ to its unit normal. By identifying tangent spaces T_pM and $T_{\nu(p)}S^2$, one obtains the *Weingarten map* $-\mathrm{d}\nu : T_pM \to T_pM$. The *second fundamental form* is given by $\Pi(w) = \langle -d\nu(w), (w)\rangle$, for any $w \in T_pM$. The determinant of the Weingarten map is the *Gauss curvature* of M. If the Gauss curvature is negative, then at any point p there will be two linearly independent vectors $v_i, i = 1, 2$ such that $\Pi(v_i) = 0$: these are the *asymptotic directions* of the surface. Any curve whose tangent at every point corresponds to an asymptotic direction is called an *asymptotic curve* or *line*. If M is pseudospherical, then M has two transverse foliations by asymptotic lines. A theorem by Beltrami–Enneper [38] states that the Gauss curvature of a surface M along an asymptotic line γ is the negative of the square of the torsion τ of γ; if M is pseudospherical, then its asymptotic lines have *constant* torsion.

Conversely, given a curve γ with constant torsion τ_0, there is a dynamical prescription for finding a pseudospherical surface M with γ as an asymptotic line:

Proposition. Let $\gamma = \gamma(s, 0)$ be the initial condition for the 'trigonometric equation' $\gamma_t = W = \cos(\theta)T - \sin(\theta)N$, where $\theta = \int^s \kappa(u)\,\mathrm{d}u$. The resulting swept-out surface M is pseudospherical with curvature $G = -\tau_0^2$. For any t, $\gamma(s, t)$ is an asymptotic curve for M. The induced evolution of θ is given by the sine–Gordon equation $\theta_{st} = -G\sin(\theta)$.

For a discussion and proof, see [31, 39].

4. Planar-like solitons and pseudospherical surfaces

4.1. Planar and planar-like solitons

As stated above, the odd functionals for LIE are restricted to planar curves. Let J_n denote the restriction of I_{2n+1} to planar curves; such functionals depend upon curvature only. A *planar soliton* is a planar curve which is a critical point for a linear combination of the J_n. For example, critical points for $J_1 + aJ_0 = \int_\gamma (\frac{1}{2}\kappa^2 + a)\,\mathrm{d}s$ have curvature functions satisfying the Euler–Lagrange equation

$$\kappa'' + \tfrac{1}{2}\kappa^3 - a\kappa = 0.$$

J_1 represents the elastic energy for a curve, J_0 a length constraint; the associated critical points are called *planar elastic curves* or *elastica.*

For simplicity, we specify boundary conditions for *asymptotic linearity* on our curves by assuming that κ and its derivatives vanish as $s \to \pm\infty$. For each J_n, we denote its associated Euler operator by E_n; the first three are

$$E_0(\kappa) = -\kappa(s)$$

$$E_1(\kappa) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa(s) + \frac{\kappa(s)^3}{2}$$

$$E_2(\kappa) = -\frac{\mathrm{d}^4}{\mathrm{d}s^4}\kappa(s) - \frac{5}{2}\frac{\kappa(s)\mathrm{d}\kappa(s)^2}{\mathrm{d}s} - \frac{5}{2}\frac{\kappa(s)^2\mathrm{d}^2\kappa(s)}{\mathrm{d}s^2} - \frac{3}{8}\kappa(s)^5.$$

A *planar-like soliton* is a space curve γ with constant torsion $\tau = \tau_0$ and for which the curvature κ is the same as that of a planar soliton. Thus, $\sum_{i=0}^n a_i E_i(\kappa) = 0$ for some choice of constants a_i. This shows that planar-like solitons are *related* to the critical points of the geometric functionals associated with LIE; the next proposition states that they *are* critical points for appropriate functionals:

Proposition. Let a space curve γ be a planar-like soliton with torsion τ_0 and curvature satisfying

$$\sum_{i=0}^{n} a_i E_i(\kappa) = 0.$$

Then γ is a critical point of the functional

$$\sum_{i=0}^{n} a_i \left(\sum_{j=0}^{2i} \binom{2i}{j} (-\tau_0)^j I_{2i+1-j} \right).$$

Equivalently, the vector field $\sum_0^n a_i A_{2i+1}$ vanishes along γ.

In the last proposition, the planar-like solitons are distinguished critical points in that they have constant torsion, which will not be true in general.

4.2. s-integrals

The E_i previously mentioned can be used to construct the mKdV hierarchy of integrable evolution equations via $\kappa_t = \mathrm{d}E_i(\kappa)/\mathrm{d}s$, $i \geqslant 0$. It is a part of the general theory of these equations [40] that the Euler operators $E(\kappa) = \sum_{i=0}^n a_i E_i(\kappa)$ are associated with the *s-integrals* $T_j(\kappa)$, where $E(\kappa)\mathrm{d}E_j(\kappa)/\mathrm{d}s = \mathrm{d}T_j(\kappa)/\mathrm{d}s$, $j = 0, 1, \ldots, n-1$. The T_j are polynomial expressions in κ and its derivatives. Along a planar-like soliton, we have $E(\kappa) = 0$, so $T_j = c_j$. In fact, for asymptotically linear curves, the c_j are all 0.

4.3. Definition and properties of T^*

For the rest of this section, γ will refer to a planar-like n-soliton with torsion $\tau_0 \neq 0$ and curvature κ satisfying $E(\kappa) = \sum_{i=0}^{n} a_i \tau_0{}^{-2n} E_i(\kappa) = 0$ with $a_0 \neq 0$, $a_n \neq 0$; we set $b_i = a_i \tau_0{}^{-2n}$. We define a planar-like vector field along γ (no binormal component)

$$T^* = (-1/b_0)\left(\sum_{i=0}^{n} b_i \Omega_i\right).$$

We describe the properties of the evolution equation $\gamma_t = T^*$ with our planar-like soliton γ as its initial condition—we call the reader's attention in particular to articles (iv) and (vii):

(i) *T^* is* CTP *along γ.* To see this, one proves the identity

$$T^* = \frac{-1}{b_0 \tau_0^2} \sum_{k=0}^{n+1} (b_{k-1} - 2b_k \tau_0^2 + b_{k+1} \tau_0^4) Z_k$$

thus expressing T^* in terms of the CTP vector fields Z_n. The variation in curvature associated with the evolution $\gamma_t = T^*$ is given by

$$\kappa_t = \frac{-\tau_0^2}{b_0} \sum_{k=0}^{n-1} b_{k+1} \frac{\mathrm{d}}{\mathrm{d}s} E_k$$

a combination of terms in the mKdV hierarchy.

(ii) *T^* preserves soliton type.* As indicated above, γ is a critical point for a linear combination of conserved functionals for the LIE hierarchy, distinguished by having constant torsion. Since T^* itself is a linear combination of terms from LIE, it deforms γ into another critical point; (i) shows that the deformation preserves constant torsion.

(iii) *T^* is of unit length along γ.* This is a direct consequence of $E(\kappa) = 0$.

(iv) *Geometry of the swept-out surface*: Let M be the surface swept out via the evolution $\gamma_t = T^*$. For any time t, T^* is a linear combination of the Frênet vectors T and N; hence the normal ν to the surface M is B, the binormal to the curve γ. We compute $\Pi(T) = \langle -\mathrm{d}\nu(T), T\rangle = \langle -\nabla_T B, T\rangle = \langle \tau N, T\rangle = 0$; T is an asymptotic direction for M. By the Beltrami–Enneper theorem, *M is a pseudospherical surface* with Gauss curvature $G = -\tau_0^2$. We will call a pseudospherical surface M a *soliton surface* if its asymptotic curves consist of planar-like solitons.

(v) *T^* as an asymptotic direction.* To show that T^* is another asymptotic direction for M, one needs to compute $\Pi(T^*) = \langle -\mathrm{d}\nu(T^*), T^*\rangle = \langle -\nabla_{T^*} B, T^*\rangle$. The term $\nabla_{T^*} B$ requires the variation formulae for frames derived in [30] which were mentioned above; the result is that T^* is indeed an asymptotic direction. We call T^* the *conjugate* asymptotic direction and its integral curve γ^* the *conjugate* asymptotic curve. By (iii), T^* is the unit tangent vector along γ^*. *Since M is pseudospherical, it must be the case that the torsion of γ^* is $\pm\tau_0$*; a calculation shows it to be τ_0.

(vi) *κ^* in terms of κ.* At a point p on M, the conjugate curvature of γ^* can be expressed in terms of the curvature of γ at that point:

$$\kappa^* = \frac{-\tau_0^2}{b_0} \sum_{0}^{n-1} b_{i+1} E_i(\kappa).$$

Again, the frame variation formulae of [30] are used to derive the Frênet equations for γ^* and hence κ^*.

(vii) γ^* *is a planar-like soliton*: by (v), we know that γ^* has torsion τ_0. The curvature function satisfies the equation

$$E^*(\kappa^*) = \sum_{i=0}^{n} b_i^* E_i^*(\kappa^*) = 0$$

where E_i^* denotes the Euler operator E_i, with differentiation with respect to s replaced by differentiation with respect to s^* ($= t =$ arclength along γ^*); $b_i^* = a_i^* \tau_0^{-2n}$, where $a_i^* = a_{n-i}$. γ^* *is therefore a planar-like soliton of the same order as* γ, *with 'flipped' coefficients.* The existence of a pseudospherical surface containing both γ and γ^* as asymptotic lines provides a *geometric Bäcklund transformation for planar-like solitons.* The proof requires use of the s-constants of motion of section 2, and the variation of curvature described in article (i).

(viii) T^* *and the 'trigonometric equation'.* We have already seen the connection between curves of constant torsion, pseudospherical surfaces and the unit-length vector field $W = \cos(\theta)T - \sin(\theta)N$. Let $A = \cos(\theta), B = -\sin(\theta)$. Then A, B satisfy the differential equations $\mathrm{d}A/\mathrm{d}s = \kappa(s)B$, $\mathrm{d}B/\mathrm{d}s = -\kappa(s)A$. T^* is also unit length; and $T^* = FT + GN$, where F, G are polynomial expressions in κ and its derivatives. One can check that along γ, F, G satisfy the same differential equations as A and B do; and at $s = -\infty$, their respective values agree. This shows that along planar-like solitons, the 'trigonometric' vector field can be expressed in terms of *local* quantities associated with the curve.

(ix) *The conjugate* LIE *hierarchy.* By definition, the vector field T^* can be expressed as a linear combination of the vector fields X_n. One can think of T^* as (minus) the zeroth term in the *conjugate* LIE hierarchy and ask whether the higher-order terms are also expressible in terms of the LIE hierarchy along γ. By the conjugate hierarchy we mean vector fields such as $X_1^* = \kappa^* B^* = \kappa^* B$, and so forth.

It is actually more convenient to express the relation between the vector fields A_n^* and A_n; this is essentially equivalent information since along a constant torsion curve the A_n span the same space as the X_n. Also, along planar-like solitons, we have $\sum_0^n b_i A_{2i+1} = 0$, so we need only consider the span of $A_1, \ldots, A_{2n}$ (an analogous statement holds for γ^*).

Proposition. Along a planar-like soliton γ, the n vector fields $A_{2i-1}^*, i = 1, \ldots, n$ can be expressed as a linear combination of the vector fields $A_{2i-1}, i = 1, \ldots, n$, and a similar statement holds for the $A_{2i}, i = 1, \ldots, n$. In particular

$$A_{2i-1}^* = \sum_{j=1}^{n} (\mathbf{S}^{-1}\mathbf{H}\mathbf{T})_{ij} A_{2j-1}$$

where $\mathbf{S}$ is the Toeplitz matrix

$$\mathbf{S} = \begin{pmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{n-1} & \cdots & b_1 & b_0 \end{pmatrix}$$

$\mathbf{H}$ is the Hankel matrix

$$\mathbf{H} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and $\mathbf{T}$ is the Toeplitz matrix

$$\mathbf{T}=\begin{pmatrix} b_n & 0 & \cdots & 0 \\ b_{n-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_1 & \cdots & b_{n-1} & b_n \end{pmatrix}.$$

A similar transformation exists relating the A^*_{2i} and A_{2i}: it is given by $\mathbf{S}^{-1}\mathbf{KT}$, where $\mathbf{K}$ is the 'almost Hankel' matrix

$$\mathbf{K}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & -b_1/b_0 \\ \vdots & \vdots & 0 & 1 & 0 & \vdots \\ \vdots & . \cdot^{\cdot} & . \cdot^{\cdot} & . \cdot^{\cdot} & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & -b_{n-2}/b_0 \\ 1 & 0 & \cdots & \cdots & 0 & -b_{n-1}/b_0 \\ 0 & \cdots & \cdots & \cdots & 0 & -b_n/b_0 \end{pmatrix}.$$

This proposition is relevant to the discussion in section 5.

(x) *Symmetry*: For a pseudospherical surface M, we have been discussing an asymptotic curve γ and its conjugate curve γ^*. Of course, there is a symmetric relation between these two curves: γ can be thought of as the conjugate curve for γ^*. The formulae we have been discussing reflect this. We mention three:

$$T=(-1/b_0^*)\left(\sum_{i=0}^{n} b_i^*\Omega_i^*\right)$$

$$\kappa=\frac{-\tau_0^2}{b_0^*}\sum_0^{n-1} b_{i+1}^* E_i^*(\kappa^*)$$

$$\kappa_{s^*}=\kappa_s^*.$$

We also remark that the formulae from articles (vii) and (ix) both have an involutive nature which also reflects this symmetry.

5. Evolution equations preserving pseudospherical surfaces

5.1. Pseudosphericity-preserving deformations and CTP *vector fields*

In a recent paper, McLachlan and Segur [39] have investigated differential geometric aspects of the evolution of surfaces in R^3. In particular, they give examples of geometric evolution equations on surfaces which preserve the pseudosphericity property, which we call *pseudosphericity-preserving* evolution equations. We now describe how their examples fit quite nicely into the structure described in this paper.

As we have seen, a pseudospherical surface M comes endowed with a foliation by curves of constant torsion (the asymptotic lines). Thus, a plausible candidate for a pseudosphericity preserving vector field would be an evolution equation defined along the asymptotic lines which preserves constant torsion. As is shown in [39], such an evolution equation exists: given $M=M_0$, let the asymptotic lines evolve according to the CTP equation $\gamma_t=Z_1(\gamma)$. Then the resulting surfaces M_t are pseudospherical.

At least for soliton pseudospherical surfaces, this can easily be extended to *any* evolution from the CTP hierarchy:

Proposition. Let $M = M_0$ be a soliton pseudospherical surface. Let the asymptotic lines evolve according to $\gamma_t = Z = \sum_0^N c_i Z_i$. Then the resulting surface M_t at any time t is pseudospherical.

The proof uses the commutativity of the LIH evolution equations. Let $\gamma = \gamma(s, 0)$ be an asymptotic curve for M; by definition, γ is a planar-like soliton. The evolution $\gamma_t = T^*$, starting with γ, sweeps out M. But along γ, T^* is just a linear combination of elements from LIH. This is also true for Z. *All* the evolution equations from LIH preserve critical points for the functionals associated with LIH, including Z. Using the CTP property of Z, the deformations of γ under Z must all be planar-like solitons of the same type. Commutativity of the Z and T^* evolution equations implies that any time t, $\gamma(s, t)$ is an asymptotic curve for the surface M_t, which is therefore pseudospherical.

5.2. *Pseudo-sphericity preserving vector fields of mixed type*

In the previous section, the deformations of the pseudospherical surface M were defined in terms of the evolution of its asymptotic line foliation. One could also define an evolution in terms of the conjugate line foliation, as well as evolutions which combine the two: $\gamma_t = Z + Z^* = \sum_0^N c_i Z_i + \sum_0^N c_i^* Z_i^*$. In [39], McLachlan and Segur essentially ask if evolution equations of this type are integrable. Using (ix) from the previous section, we can answer in the affirmative, again assuming M is a soliton surface. The reasoning is simple: along such a surface, the A_i^*, and therefore the Z_i^*, can be expressed in terms of the A_i. In fact, one checks that the Z_i^* are linear combinations of the Z_i, hence the second summand is redundant.

Acknowledgments

We have already cited the paper by Melko and Sterling [15], which provides an alternative approach to studying pseudospherical surfaces. It was their work which suggested to us the connection between pseudospherical surfaces and LIH; we refer the reader to that paper and, in particular, its interesting and suggestive computer graphics of pseudospherical surfaces.

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